

Variational Inequalities of Navier–Stokes Type with Time Dependent Constraints

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Abstract. We consider a class of parabolic variational inequalities with time dependent obstacle of the form $|\mathbf{u}(x, t)| \leq p(x, t)$, where \mathbf{u} is the velocity field of a fluid governed by the Navier–Stokes variational inequality. The obstacle function $p = p(x, t)$ imposed on \mathbf{u} consists of three parts which are respectively the degenerate part $p(x, t) = 0$, the finitely positive part $0 < p(x, t) < \infty$ and singular part $p(x, t) = \infty$. In this paper, we shall propose a sequence of approximate obstacle problems with everywhere finitely positive obstacles and prove an existence result for the original problem by discussing the convergence of the approximate problems. The crucial step is to handle the nonlinear convection term. In this paper we propose a new approach to it.

1 Introduction

In real problems, we find many dynamical processes which occur in fluids or in consequence of a fluid flow. Their mathematical models include then a hydrodynamic equation, typically Stokes or Navier–Stokes, coupled with some other evolution systems, such as heat convection [14, 16], phase transitions [1] or biofilm growth [11, 23]. These couplings may have the form of transport or advection, but they may also mean some evolution of the domain in which the flow takes place. To give just one example of phenomenon of importance to medicine and in which both types of couplings appear at the same time scale, let us have a look on the mentioned biomass growth. In a fluid transporting some living organisms and some appropriate nutrient, some of these organisms can stick to the boundary of the fluid flow’s domain (e.g. blood vessels walls) and then aggregate, in which way they gradually restrict the domain available for the flow, forming a geometrical obstacle to it.

Mathematical analysis of models for such systems seems not easy, since the theory on partial differential systems coupled with equations, or variational inequalities, of the Navier–Stokes type has not been completely established.

In this paper, we address the problem of a Navier–Stokes flow constrained by some evolving in time obstacle. We model the obstacle as a non-negative function p , depending on the space and time variable, which is a bound imposed *a priori* on the velocity of the flow. The Navier–Stokes equation becomes then naturally a variational inequality. We allow the constraint to disappear ($p = \infty$, free flow), to be a total obstacle ($p = 0$, no flow) or only partial ($0 < p < \infty$). We assume that p is continuous. Our main result is Theorem 1.1 below, stating existence and some regularity of solution to this problem.

This kind of parabolic obstacle problem would be useful for mathematical modelling of various nonlinear problems in hydrodynamic fluids, see e.g. [2, 9, 10, 11, 18, 19, 21]. As far as variational inequalities of Navier–Stokes type are concerned, see e.g. [3–6, 24, 25] for a constant in time constraint, and [13], where the constraint can be time and space dependent. However, even this last case did not allow the ”free flow” and ”no-flow” regions, i.e. the obstacle function p had to be finite and bounded from below by a positive constant — a serious limitation of the model that we overcome in the present work. It is clear that especially allowing the ”total obstacle” case, i.e. having regions where $p = 0$, is essential from the point of view of modelling; it is also the main challenge for the mathematical analysis that we are presenting.

For basic studies on Navier–Stokes equations and phase transitions, we refer to [27] and [8], respectively. Our formulation of the Navier–Stokes inequality arising from the obstacle, that we state in Definition 1.1 below, is analogous to these appearing in [3–6, 24, 25]. For its analysis, we will use the theory of subdifferentials contained in [7, 17, 22, 28]. This will be exposed in Section 2.

Let us set the basic functional framework and explicit the assumptions so as to formulate the main result. Let Ω be a bounded domain in \mathbf{R}^3 with smooth boundary $\Gamma := \partial\Omega$, $Q := \Omega \times (0, T)$, $0 < T < \infty$ and $\Sigma := \Gamma \times (0, T)$, and denote by $|\cdot|_X$ the norm in various function spaces X built on Ω as well as by $\|\cdot\|_Y$ for function spaces Y on $\Omega \times (0, T)$. Also, consider the usual solenoidal function spaces:

$$\begin{aligned}\mathcal{D}_\sigma(\Omega) &:= \{\mathbf{v} = (v^{(1)}, v^{(2)}, v^{(3)}) \in \mathcal{D}(\Omega)^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}, \\ \mathbf{H}_\sigma(\Omega) &:= \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega)^3, \text{ with norm } |\cdot|_{0,2}, \\ \mathbf{V}_\sigma(\Omega) &:= \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } H_0^1(\Omega)^3, \text{ with norm } |\cdot|_{1,2}, \\ \mathbf{W}_\sigma(\Omega) &:= \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } W_0^{1,4}(\Omega)^3, \text{ with norm } |\cdot|_{1,4};\end{aligned}$$

in these spaces the norms are given as usual by

$$|\mathbf{v}|_{0,2} := \left\{ \sum_{k=1}^3 \int_{\Omega} |v^{(k)}|^2 dx \right\}^{\frac{1}{2}}, \quad |\mathbf{v}|_{1,2} := \left\{ \sum_{k=1}^3 \int_{\Omega} |\nabla v^{(k)}|^2 dx \right\}^{\frac{1}{2}}$$

and

$$|\mathbf{v}|_{1,4} := \left\{ \sum_{k=1}^3 \int_{\Omega} |\nabla v^{(k)}|^4 dx \right\}^{\frac{1}{4}}.$$

For simplicity we denote the dual spaces of $\mathbf{V}_\sigma(\Omega)$ and $\mathbf{W}_\sigma(\Omega)$ by $\mathbf{V}_\sigma^*(\Omega)$ and $\mathbf{W}_\sigma^*(\Omega)$, respectively, which are equipped with their dual norms $|\cdot|_{-1,2}$ and $|\cdot|_{-1,\frac{4}{3}}$. Also, we denote the inner product in $\mathbf{H}_\sigma(\Omega)$ by $(\cdot, \cdot)_\sigma$ and the duality between $\mathbf{V}_\sigma^*(\Omega)$ and $\mathbf{V}_\sigma(\Omega)$ by $\langle \cdot, \cdot \rangle_\sigma$, namely for $\mathbf{v}_i = (v_i^{(1)}, v_i^{(2)}, v_i^{(3)})$, $i = 1, 2$,

$$(\mathbf{v}_1, \mathbf{v}_2)_\sigma := \sum_{k=1}^3 \int_{\Omega} v_1^{(k)} v_2^{(k)} dx, \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\sigma = \sum_{k=1}^3 \int_{\Omega} \nabla v_1^{(k)} \cdot \nabla v_2^{(k)} dx.$$

Then, by identifying the dual of $\mathbf{H}_\sigma(\Omega)$ with itself, we have:

$$\mathbf{V}_\sigma(\Omega) \hookrightarrow \mathbf{H}_\sigma(\Omega) \hookrightarrow \mathbf{V}_\sigma^*(\Omega), \quad \mathbf{W}_\sigma(\Omega) \hookrightarrow C(\overline{\Omega})^3;$$

and all these embeddings are compact.

We are given a non-negative function $p = p(x, t)$ on \overline{Q} as an obstacle function such that $0 \leq p(x, t) \leq \infty$ for all $(x, t) \in \overline{Q}$ and p is continuous from \overline{Q} into $[0, \infty]$, namely,

$$\left\{ \begin{array}{l} \text{the set } \overline{Q}(p = \infty) := \{(x, t) \in \overline{Q} \mid p(x, t) = \infty\} \text{ is closed in } \overline{Q}, \\ \forall \kappa \in (0, \infty), p \text{ is continuous on } \overline{Q}(p \leq \kappa) := \{(x, t) \in \overline{Q} \mid p(x, t) \leq \kappa\}, \\ \forall M \in (0, \infty), \text{ there is an open set } U_M \text{ containing } \overline{Q}(p = \infty) \\ \text{such that } p \geq M \text{ on } U_M \cap \overline{Q}. \end{array} \right. \quad (1.1)$$

It is easily seen that (1.1) is equivalent to the continuity on \overline{Q} in the usual sense, of the function

$$\alpha(x, t) := \begin{cases} \frac{p(x, t)}{1 + p(x, t)}, & \text{if } 0 \leq p(x, t) < \infty, \\ 1. & \text{if } p(x, t) = \infty, \end{cases}$$

We are now ready to define the solution of our obstacle problem.

Definition 1.1. *For given data*

$$\nu > 0 \text{ (constant)}, \mathbf{g} \in L^2(0, T; \mathbf{H}_\sigma(\Omega)), \mathbf{u}_0 \in \mathbf{H}_\sigma(\Omega),$$

our obstacle problem $P(p; \mathbf{g}, \mathbf{u}_0)$ is to find a solution $\mathbf{u} := (u^{(1)}, u^{(2)}, u^{(3)})$ from $[0, T]$ into $\mathbf{H}_\sigma(\Omega)$ satisfying the following (i) and (ii):

(i) $\mathbf{u}(0) = \mathbf{u}_0$ in $\mathbf{H}_\sigma(\Omega)$, and $t \mapsto (\mathbf{u}(t), \mathbf{v}(t))_\sigma$ is of bounded variation on $[0, T]$ for all $\mathbf{v} \in \mathcal{K}(p)$, where

$$\mathcal{K}(p) := \left\{ \mathbf{v} \in C^1([0, T]; \mathbf{W}_\sigma(\Omega)) \mid \begin{array}{l} |\mathbf{v}| \leq p \text{ on } Q, \text{ supp}(\mathbf{v}) \text{ is compact} \\ \text{in } \{(x, t) \in \Omega \times [0, T] \mid p(x, t) > 0\} \end{array} \right\},$$

and $\text{supp}(\mathbf{v})$ denotes the support of \mathbf{v} ,

(ii) $\mathbf{u} : [0, T] \rightarrow \mathbf{H}_\sigma(\Omega)$, $\sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{0,2} < \infty$, $\mathbf{u} \in L^2(0, T; \mathbf{V}_\sigma(\Omega))$ and

$$\begin{aligned} & |\mathbf{u}(x, t)| \leq p(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \in [0, T], \\ & \int_0^t (\mathbf{v}'(\tau), \mathbf{u}(\tau) - \mathbf{v}(\tau))_\sigma d\tau + \nu \int_0^t \langle \mathbf{u}(\tau), \mathbf{u}(\tau) - \mathbf{v}(\tau) \rangle_\sigma d\tau \\ & + \int_0^t \int_\Omega (\mathbf{u}(x, \tau) \cdot \nabla) \mathbf{u}(x, \tau) \cdot \nabla (\mathbf{u}(x, \tau) - \mathbf{v}(x, \tau)) dx d\tau + \frac{1}{2} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{0,2}^2 \\ & \leq \int_0^t (\mathbf{g}(\tau), \mathbf{u}(\tau) - \mathbf{v}(\tau))_\sigma d\tau + \frac{1}{2} \|\mathbf{u}_0 - \mathbf{v}(0)\|_{0,2}^2, \quad \forall t \in [0, T], \quad \forall \mathbf{v} \in \mathcal{K}(p). \end{aligned} \quad (1.2)$$

We note that \mathbf{u} is defined for every $t \in [0, T]$, and according to the given \mathbf{u}_0 , even if we do not require it to be continuous in time: our definition permits jumps in time, including the initial time $t = 0$. What we will prove, is that \mathbf{u} is a limit of continuous approximate solutions.

The main objective of this paper is to prove the following existence result for $P(p; \mathbf{g}, \mathbf{u}_0)$.

Theorem 1.1. *Assume that (1.1) is satisfied and*

$$\mathbf{u}_0 \in \mathbf{W}_\sigma(\Omega), \quad \text{supp}(\mathbf{u}_0) \subset \{x \in \Omega \mid p(x, 0) > 0\}, \quad |\mathbf{u}_0| \leq p(\cdot, 0) \text{ in } \Omega. \quad (1.3)$$

Then, there is at least one solution \mathbf{u} of $P(p; \mathbf{g}, \mathbf{u}_0)$.

We do not touch the uniqueness of solution problem, even if uniqueness holds for constraints considered in [3–6, 13, 24, 25]. In our case it remains an open question, together with time continuity. We state uniqueness for approximate solutions, defined in Section 2 (see Proposition 2.1).

In the proof of Theorem 1.1 the main difficulty arises from the nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$. In our case, the class of test functions $\mathcal{K}(p)$ is not a linear space. Therefore, the usual compactness methods, based on Sobolev embeddings, cannot be directly applied. Our idea is to use local bounded variation estimate of \mathbf{u} or its approximate solutions in the space $(0, T) \times \mathbf{W}_\sigma^*(\Omega)$; see Section 3. This is a completely new approach to parabolic variational inequalities of the Navier–Stokes type. The proof of Theorem 1.1 is given in Section 4.

2 Approximate problems

In this section, we propose an approximation procedure to $P(p; \mathbf{g}, \mathbf{u}_0)$. We begin with a regular approximation of the obstacle function $p(x, t)$. Choose a sequence $\{p_n\}$ of Lipschitz, non-degenerate obstacle functions on \overline{Q} such that

$$\left\{ \begin{array}{l} 0 < p_n(x, t) < \infty \text{ on } \overline{Q}, \forall n \in \mathbf{N}, \\ \forall \kappa \in (0, \infty), p_n \xrightarrow{n \rightarrow \infty} p \text{ uniformly in } \overline{Q}(p \leq \kappa) := \{(x, t) \in \overline{Q} \mid p(x, t) \leq \kappa\}, \\ \text{for any sufficiently large } M > 0, \text{ there is an integer } n_M \in \mathbf{N} \text{ such that} \\ M \leq p_n \leq p \text{ on } \overline{Q}(p > M) := \{(x, t) \in \overline{Q} \mid p(x, t) > M\}, \forall n \geq n_M. \end{array} \right. \quad (2.1)$$

Remark 2.1. Given function $p(x, t)$ satisfying (1.1), we always construct an approximate sequence $\{p_n\}$ satisfying (2.1). For instance, the sequence $\{p_n\}$ consisting of regularizations of cut-off functions

$$\tilde{p}_n(x, t) := \begin{cases} n, & \text{if } p(x, t) > n, \\ p(x, t), & \text{if } \frac{1}{n} \leq p(x, t) \leq n, \\ \frac{1}{n}, & \text{if } 0 \leq p(x, t) < \frac{1}{n}, \end{cases}$$

fulfills (2.1).

Next, we formulate precisely the approximate problem, denoted by $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$.

Definition 2.1 *Given a function p_n satisfying (2.1) and an initial datum*

$$\mathbf{u}_{0n} \in \mathbf{V}_\sigma(\Omega), \quad |\mathbf{u}_{0n}| \leq p_n(\cdot, 0) \text{ a.e. in } \Omega, \quad (2.2)$$

the problem $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$ is to find a function $\mathbf{u}_n = (u_n^{(1)}, u_n^{(2)}, u_n^{(3)})$ which satisfies the following (1) and (2):

(1) $\mathbf{u}_n \in W^{1,2}(0, T; \mathbf{H}_\sigma(\Omega)) \cap C([0, T]; \mathbf{V}_\sigma(\Omega))$ such that

$$|\mathbf{u}_n(x, t)| \leq p_n(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \in [0, T], \quad (2.3)$$

$$\begin{aligned} & (\mathbf{u}'_n(t), \mathbf{u}_n(t) - \mathbf{z})_\sigma + \nu \langle \mathbf{u}_n(t), \mathbf{u}_n(t) - \mathbf{z} \rangle_\sigma \\ & + \int_\Omega (\mathbf{u}_n(t) \cdot \nabla) \mathbf{u}_n(t) \cdot (\mathbf{u}_n(t) - \mathbf{z}) dx \leq (\mathbf{g}(t), \mathbf{u}_n(t) - \mathbf{z})_\sigma, \\ & \forall \mathbf{z} \in \mathbf{V}_\sigma(\Omega) \quad \text{with } |\mathbf{z}| \leq p_n(\cdot, t) \quad \text{a.e. in } \Omega, \quad \text{a.e. } t \in [0, T]. \end{aligned} \quad (2.4)$$

(2) $\mathbf{u}_n(0) = \mathbf{u}_{0n}$ in $\mathbf{H}_\sigma(\Omega)$.

We are now applying the general theory on evolution inclusions generated by time dependent subdifferentials to the solvability of $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$. To this end, we introduce a time dependent convex function φ_n^t , $t \in [0, T]$, on $\mathbf{H}_\sigma(\Omega)$ given by:

$$\varphi_n^t(\mathbf{z}) := \begin{cases} \frac{\nu}{2} |\mathbf{z}|_{1,2}^2 + I_{K(p_n;t)}(\mathbf{z}), & \forall \mathbf{z} \in \mathbf{V}_\sigma(\Omega), \\ \infty, & \text{otherwise,} \end{cases}$$

where

$$K(p_n; t) := \{\mathbf{z} \in \mathbf{V}_\sigma(\Omega) \mid |\mathbf{z}| \leq p_n(\cdot, t) \quad \text{a.e. in } \Omega\},$$

which is closed and convex in $\mathbf{V}_\sigma(\Omega)$, and $I_{K(p_n;t)}$ is its indicator function on $\mathbf{V}_\sigma(\Omega)$, namely

$$I_{K(p_n;t)}(\mathbf{z}) := \begin{cases} 0, & \text{if } \mathbf{z} \in K(p_n; t), \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly φ_n^t is non-negative, proper, l.s.c. and convex on $\mathbf{H}_\sigma(\Omega)$ and on $\mathbf{V}_\sigma(\Omega)$ for every $t \in [0, T]$. Also, we define a perturbation term $\mathbf{G}(t, \cdot) : K(p_n; t) \rightarrow \mathbf{H}_\sigma(\Omega)$ by the formula:

$$(\mathbf{G}(t, \mathbf{v}), \mathbf{z})_\sigma := \int_\Omega (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{z} dx = \sum_{k,j=1}^3 \int_\Omega v^{(k)}(x) \frac{\partial v^{(j)}(x)}{\partial x_k} z^{(j)}(x) dx$$

for all $\mathbf{v} = (v^{(1)}, v^{(2)}, v^{(3)}) \in K(p_n; t)$ and $\mathbf{z} = (z^{(1)}, z^{(2)}, z^{(3)}) \in \mathbf{H}_\sigma(\Omega)$.

Lemma 2.1. *Let \mathbf{u}_n be a function in $W^{1,2}(0, T; \mathbf{H}_\sigma(\Omega)) \cap C([0, T]; \mathbf{V}_\sigma(\Omega))$ and let $\mathbf{u}_{0n} \in K(p_n; 0)$. Then, $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$ is equivalent to the following Cauchy problem:*

$$\begin{cases} \mathbf{u}'_n(t) + \partial \varphi_n^t(\mathbf{u}_n(t)) + \mathbf{G}(t, \mathbf{u}_n(t)) \ni \mathbf{g}(t) & \text{in } \mathbf{H}_\sigma(\Omega), \quad \text{a.e. } t \in [0, T], \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \end{cases} \quad (2.5)$$

where $\partial \varphi_n^t(\cdot)$ is the subdifferential of $\varphi_n^t(\cdot)$ in $\mathbf{H}_\sigma(\Omega)$.

The equivalence required in Lemma 2.1 is derived immediately from the definition of the subdifferential $\partial \varphi_n^t$, namely $\mathbf{v}^* \in \partial \varphi_n^t(\mathbf{v})$ if and only if $\mathbf{v}^* \in \mathbf{H}_\sigma(\Omega)$, $\mathbf{v} \in K(p_n; t)$ and

$$(\mathbf{v}^*, \mathbf{z} - \mathbf{v})_\sigma + \nu \langle \mathbf{v}, \mathbf{v} - \mathbf{z} \rangle_\sigma \leq 0, \quad \forall \mathbf{z} \in K(p_n, t).$$

For a detailed proof, see [13, 14].

Lemma 2.2. *Let p_n satisfy (2.1). There is a positive constant C_n , depending on n , such that for every $s, t \in [0, T]$ and every $\mathbf{z} \in K(p_n; s)$ there is $\tilde{\mathbf{z}} \in K(p_n; t)$ satisfying*

$$|\tilde{\mathbf{z}} - \mathbf{z}|_{0,2} \leq C_n |p_n(t) - p_n(s)|_{C(\bar{\Omega})}, \quad \varphi_n^t(\tilde{\mathbf{z}}) \leq \varphi_n^s(\mathbf{z}). \quad (2.6)$$

Proof. Let $\mu_n = \min_{(x,t) \in \bar{Q}} p_n(x, t)$; we note that $\mu_n > 0$ by (2.1). Denote also by L_n the Lipschitz constant of p_n and choose a partition of $[0, T]$, $0 = t_0 < t_1 < t_2 < \dots < t_N = T$, so that $t_k - t_{k-1} \leq \mu_n / L_n$ for $k = 1, \dots, N$. Then

$$|p_n(t) - p_n(s)| < \mu_n, \quad \text{for } t, s \in [t_k, t_{k+1}], \quad k = 1, 2, \dots, N.$$

Now, suppose that $s, t \in [t_k, t_{k+1}]$. Given $\mathbf{z} \in K(p_n; s)$, we consider the function

$$\tilde{\mathbf{z}}(x) := \left(1 - \frac{1}{\mu_n} |p_n(s) - p_n(t)|_{C(\bar{\Omega})}\right) \mathbf{z}(x).$$

Observe that

$$\begin{aligned} |\tilde{\mathbf{z}}(x)| &= \left(1 - \frac{1}{\mu_n} |p_n(s) - p_n(t)|_{C(\bar{\Omega})}\right) |\mathbf{z}(x)| \\ &\leq \left(1 - \frac{1}{\mu_n} |p_n(s) - p_n(t)|_{C(\bar{\Omega})}\right) p_n(x, s) \\ &\leq p_n(x, s) - |p_n(s) - p_n(t)|_{C(\bar{\Omega})} \\ &\leq p_n(x, s) - |p_n(x, s) - p_n(x, t)| \leq p_n(x, t). \end{aligned}$$

Since $\operatorname{div} \tilde{\mathbf{z}} = 0$, the above inequality implies that $\tilde{\mathbf{z}} \in K(p_n; t)$. Clearly, the second inequality of (2.6) is satisfied. Moreover,

$$|\tilde{\mathbf{z}} - \mathbf{z}|_{0,2} = \frac{1}{\mu_n} |p_n(s) - p_n(t)|_{C(\bar{\Omega})} |\mathbf{z}|_{0,2} \leq C'_n |p_n(t) - p_n(s)|_{C(\bar{\Omega})}$$

where

$$C'_n = \frac{\max_{(x,t) \in \bar{Q}} p_n(x, t)}{\mu_n} |\Omega|^{\frac{1}{2}},$$

which is finite by (2.1).

In the general case of $s, t \in [0, T]$ and $\mathbf{z} \in K(p_n; s)$, by repeating the above procedures at most N -times, we can construct $\tilde{\mathbf{z}} \in K(p_n; t)$ satisfying both required inequalities, the first one with the constant $C_n := NC'_n$. \square

We prepare a lemma which we shall need in the convergence of approximate problems.

Lemma 2.3. *Let p_n satisfy (2.1). Let \mathbf{v} be any function in $C([0, T]; \mathbf{W}_\sigma(\Omega))$ such that*

$$\operatorname{supp}(\mathbf{v}) \subset \{(x, t) \in \Omega \times [0, T] \mid p(x, t) \geq \delta\}, \quad |\mathbf{v}| \leq p \quad \text{on } Q$$

for a positive number δ , and put

$$\delta_n := \frac{1}{\delta} \max_{\operatorname{supp}(\mathbf{v})} |p \wedge M - p_n \wedge M|, \quad \forall n \in \mathbf{N},$$

where $M = \delta + \sup_{(x,t) \in Q} |\mathbf{v}(x,t)|$, $p \wedge M = \min\{p, M\}$ and $p_n \wedge M = \min\{p_n, M\}$. Then $\delta_n \rightarrow 0$ and for any $n \in \mathbf{N}$, we have

$$(1 - \delta_n)^+ \mathbf{v}(t) \in K(p_n; t), \quad \forall t \in [0, T]. \quad (2.7)$$

Proof. It follows easily from (2.1) that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. In addition, in case $\delta_n < 1$,

$$\begin{aligned} (1 - \delta_n)^+ |\mathbf{v}(x, t)| &\leq (1 - \delta_n) |\mathbf{v}(x, t)| \leq (1 - \delta_n) p(x, t) \wedge M \\ &= p(x, t) \wedge M - \frac{p(x, t) \wedge M}{\delta} |p(x, t) \wedge M - p_n(x, t) \wedge M| \\ &\leq p(x, t) \wedge M - (p(x, t) \wedge M - p_n(x, t) \wedge M) \leq p_n(x, t) \end{aligned}$$

for all $(x, t) \in \text{supp}(\mathbf{v})$. This shows (2.7). \square

Proposition 2.1. *Let p_n satisfy (2.1). Let \mathbf{u}_0 be any element in $\mathbf{W}_\sigma(\Omega)$ for which (1.3) holds; hence*

$$\text{supp}(\mathbf{u}_0) \subset \{x \in \Omega \mid p(x, 0) \geq \hat{\delta}\}$$

for a certain constant $\hat{\delta} > 0$. Put

$$\mathbf{u}_{0n} := (1 - \hat{\delta}_n)^+ \mathbf{u}_0 \quad \text{with} \quad \hat{\delta}_n := \frac{1}{\hat{\delta}} \max_{\text{supp}(\mathbf{u}_0)} |p \wedge \hat{M} - p_n \wedge \hat{M}|, \quad \forall n \in \mathbf{N},$$

where $\hat{M} = \hat{\delta} + |\mathbf{u}_0|_{C(\overline{\Omega})}$. Then problem $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$ admits one and only one solution \mathbf{u}_n in $W^{1,2}(0, T; \mathbf{H}_\sigma(\Omega)) \cap C([0, T]; \mathbf{V}_\sigma(\Omega))$, which is also the unique solution of (2.5). Moreover, \mathbf{u}_n satisfies the estimate

$$|\mathbf{u}_n(t)|_{0,2}^2 + \nu \int_0^t |\mathbf{u}_n|_{1,2}^2 d\tau \leq |\mathbf{u}_0|_{0,2}^2 + \frac{L_P^2}{\nu} \int_0^t |\mathbf{g}|_{0,2}^2 d\tau =: M_0, \quad \forall t \in [0, T], \quad (2.8)$$

where L_P is the Poincaré constant, i.e.

$$|\mathbf{z}|_{0,2} \leq L_P |\mathbf{z}|_{1,2}, \quad \forall \mathbf{z} \in \mathbf{V}_\sigma.$$

Proof. We show in a similar way to that of Lemma 2.3 that \mathbf{u}_{0n} satisfies (2.2) for all n . The time dependence (2.6) of the mapping $t \mapsto \varphi_n^t$ is a sufficient condition for the Cauchy problem (2.5) without perturbation \mathbf{G} to have one and only one solution \mathbf{u} (see [17, 22, 28]). Furthermore, as to the perturbation term \mathbf{G} , we have

$$|(\mathbf{G}(t, \mathbf{v}) - \mathbf{G}(t, \bar{\mathbf{v}}), \mathbf{v} - \bar{\mathbf{v}})_\sigma| \leq \varepsilon |\mathbf{v} - \bar{\mathbf{v}}|_{1,2}^2 + C_\varepsilon |\mathbf{v} - \bar{\mathbf{v}}|_{0,2}^2, \quad \forall \mathbf{v}, \bar{\mathbf{v}} \in K(p_n; t), \quad (2.9)$$

where ε is any positive constant and C_ε is a positive constant depending only on ε and n . Indeed,

$$\begin{aligned} (\mathbf{G}(t, \mathbf{v}) - \mathbf{G}(t, \bar{\mathbf{v}}), \mathbf{v} - \bar{\mathbf{v}})_\sigma &= \sum_{k,j=1}^3 \int_\Omega \left(v^{(k)} \frac{\partial v^{(j)}}{\partial x_k} - \bar{v}^{(k)} \frac{\partial \bar{v}^{(j)}}{\partial x_k} \right) (v^{(j)} - \bar{v}^{(j)}) dx \\ &= \sum_{k,j=1}^3 \int_\Omega (v^{(k)} - \bar{v}^{(k)}) \frac{\partial v^{(j)}}{\partial x_k} (v^{(j)} - \bar{v}^{(j)}) dx + \sum_{k,j=1}^3 \int_\Omega \bar{v}^{(k)} \frac{\partial (v^{(j)} - \bar{v}^{(j)})}{\partial x_k} (v^{(j)} - \bar{v}^{(j)}) dx, \end{aligned}$$

and from the fact that $\operatorname{div} \mathbf{v} = \operatorname{div} \bar{\mathbf{v}} = 0$ we infer that the second sum is equal to 0, while the first is estimated by $9(\max_{\Omega} |\mathbf{v}|) |\mathbf{v} - \bar{\mathbf{v}}|_{1,2} |\mathbf{v} - \bar{\mathbf{v}}|_{0,2}$, so that (2.9) follows. Therefore, according to the perturbation result of [26], $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$ has a unique solution \mathbf{u}_n . Also, by taking $\mathbf{z} = 0$ in (2.4) and integrating in time over $[0, t]$ we get

$$\frac{1}{2} |\mathbf{u}_n(t)|_{0,2}^2 + \nu \int_0^t |\mathbf{u}_n|_{1,2}^2 d\tau \leq \frac{1}{2} |\mathbf{u}_{0n}|_{0,2}^2 + \int_0^t (\mathbf{g}, \mathbf{u}_n)_{\sigma} d\tau. \quad (2.10)$$

Noting that

$$\int_0^t |(\mathbf{g}, \mathbf{u}_n)_{\sigma}| d\tau \leq \frac{\nu}{2} \int_0^t |\mathbf{u}_n|_{1,2}^2 d\tau + \frac{L_P^2}{2\nu} \int_0^t |\mathbf{g}|_{0,2}^2 d\tau,$$

we obtain (2.8) from (2.10), since $|\mathbf{u}_{0n}|_{0,2} \leq |\mathbf{u}_0|_{0,2}$. \square

It follows from the energy estimate (2.8) that there exists a subsequence $\{\mathbf{u}_{n_k}\}$ and a function $\mathbf{u} \in L^2(0, T; \mathbf{V}_{\sigma}(\Omega))$ such that \mathbf{u}_{n_k} weakly converges to \mathbf{u} in $L^2(0, T; \mathbf{V}_{\sigma}(\Omega))$ as well as weakly* in $L^{\infty}(0, T; \mathbf{H}_{\sigma}(\Omega))$ (as $k \rightarrow \infty$). In the sequel, for simplicity of notation we write $\{\mathbf{u}_n\}$ again for $\{\mathbf{u}_{n_k}\}$, namely

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; \mathbf{V}_{\sigma}(\Omega)) \text{ and weakly}^* \text{ in } L^{\infty}(0, T; \mathbf{H}_{\sigma}(\Omega)) \text{ as } n \rightarrow \infty. \quad (2.11)$$

We will refer to (2.11) in most statements and proofs which follow. We underline however that we have here only a subsequence of the sequence \mathbf{u}_n constructed in Proposition 2.1.

3 Local uniform estimate of the total variation of \mathbf{u}_n

In this section we use the notation:

$$\begin{aligned} \hat{Q} &:= \Omega \times [0, T], \\ \hat{Q}(p > \kappa) &:= \{(x, t) \in \hat{Q} \mid p(x, t) > \kappa\}, \quad 0 < \kappa < \infty, \\ \hat{Q}(p = \infty) &:= \{(x, t) \in \hat{Q} \mid p(x, t) = \infty\}, \\ \hat{Q}(p = 0) &:= \{(x, t) \in \hat{Q} \mid p(x, t) = 0\}; \end{aligned}$$

by (1.1), $\hat{Q}(p > \kappa)$ is relatively open in \hat{Q} , and $\hat{Q}(p = \infty)$ and $\hat{Q}(p = 0)$ are relatively compact in \hat{Q} . We will also use in this section the spaces $\mathbf{W}_{\sigma}(\Omega')$, $\mathbf{V}_{\sigma}(\Omega')$, $\mathbf{H}_{\sigma}(\Omega')$, built on any open set $\Omega' \subset \Omega$, and use the same notation as in Section 1 for the norms without indicating Ω' explicitly therein.

We shall use the continuous embeddings:

$$\mathbf{W}_{\sigma}(\Omega') \hookrightarrow C(\overline{\Omega'})^3, \quad \mathbf{W}_{\sigma}(\Omega') \hookrightarrow \mathbf{V}_{\sigma}(\Omega') \hookrightarrow L^4(\Omega')^3 \hookrightarrow L^2(\Omega')^3,$$

with inequalities

$$\begin{aligned} |\mathbf{f}|_{C(\overline{\Omega'})^3} &\leq L_0 |\mathbf{f}|_{1,4}, \quad |\mathbf{f}|_{1,2} \leq L_2 |\mathbf{f}|_{1,4}, \quad \forall \mathbf{f} \in \mathbf{W}_{\sigma}(\Omega'), \\ |\mathbf{f}|_{0,2} &\leq L_1 |\mathbf{f}|_{1,2}, \quad |\mathbf{f}|_{0,4} := |\mathbf{f}|_{L^4(\Omega')^3} \leq L_3 |\mathbf{f}|_{1,2}, \quad \forall \mathbf{f} \in \mathbf{V}_{\sigma}(\Omega'), \end{aligned} \quad (3.1)$$

where L_0 , $L_1 := L_P$, L_2 and L_3 are positive constants, which are derived from the Sobolev inequalities as well as the Poincaré inequality, and independent of $\Omega' \subset \Omega$ (cf. [15; Ch.7]).

Lemma 3.1. *Let p satisfy (1.1), $\kappa > 0$, Ω' be an open set in Ω , and $\hat{Q}' := \Omega' \times [T_1, T_1']$. Assume that $\tilde{\Omega}' \times [T_1, T_1'] \subset \hat{Q}'(p > \kappa)$, where $\tilde{\Omega}'$ is the relative closure of Ω' in Ω . Then, for n large enough, \mathbf{u}_n defined in Proposition 2.1 is of bounded variation as a function from $[T_1, T_1']$ into $\mathbf{W}_\sigma^*(\Omega')$. Its total variation is uniformly bounded with respect to n and the bound depends only on κ .*

Proof. By (2.1), we fix $N = N(\kappa)$ large enough to have

$$\frac{\kappa}{2} \leq p_n(x, t), \quad \forall n > N, \quad \forall (x, t) \in \hat{Q}'(p > \kappa). \quad (3.2)$$

Take now $\mathbf{z} \in C([T_1, T_1']; \mathbf{W}_\sigma(\Omega'))$ with

$$\mathbf{z}(T_1) = \mathbf{z}(T_1') = 0, \quad \text{supp } (\mathbf{z}) \subset \Omega' \times [T_1, T_1'], \quad \|\mathbf{z}\|_{C([T_1, T_1']; \mathbf{W}_\sigma(\Omega'))} \leq \frac{\kappa}{2L_0}, \quad (3.3)$$

where L_0 is defined by (3.1). Then by (3.1), (3.2) and (3.3),

$$|\mathbf{z}(x, t)| \leq \|\mathbf{z}\|_{C([T_1, T_1']; C(\overline{\Omega'})^3)} \leq \frac{\kappa}{2} \leq p_n(x, t), \quad \forall (x, t) \in \hat{Q}'(p > \kappa), \quad \forall n > N.$$

So, $\mathbf{z}(\cdot, t) \in K(p_n; t)$ for $n > N$, i.e. $\mathbf{z}(t)$ is a proper test function in (2.3), which writes

$$\begin{aligned} (\mathbf{u}'_n(t), \mathbf{u}_n(t) - \mathbf{z}(t))_\sigma + \nu \langle \mathbf{u}_n(t), \mathbf{u}_n(t) - \mathbf{z}(t) \rangle_\sigma + (\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{u}_n(t) - \mathbf{z}(t))_\sigma \\ \leq (\mathbf{g}(t), \mathbf{u}_n(t) - \mathbf{z}(t))_\sigma, \quad \text{for a. e. } t \in [T_1, T_1']. \end{aligned}$$

As $(\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{u}_n(t))_\sigma = 0$, we have for a.e. $t \in [T_1, T_1']$, all $n > N$ and \mathbf{z} satisfying (3.3):

$$\begin{aligned} -(\mathbf{u}'_n(t), \mathbf{z}(t))_\sigma + \frac{1}{2} \frac{d}{dt} |\mathbf{u}_n(t)|_{0,2}^2 + \nu |\mathbf{u}_n(t)|_{1,2}^2 \\ \leq \nu |\mathbf{u}_n(t)|_{1,2} |\mathbf{z}(t)|_{1,2} + (\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{z}(t))_\sigma + (\mathbf{g}(t), \mathbf{u}_n(t) - \mathbf{z}(t))_\sigma. \end{aligned}$$

When integrated in time, with the Young, Schwarz and Poincaré inequalities, this implies:

$$\begin{aligned} - \int_{T_1}^{T_1'} (\mathbf{u}'_n(t), \mathbf{z}(t))_\sigma dt + \frac{1}{2} |\mathbf{u}_n(T_1')|_{0,2}^2 + \frac{\nu}{2} \int_{T_1}^{T_1'} |\mathbf{u}_n(t)|_{1,2}^2 dt \\ \leq \int_{T_1}^{T_1'} (\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{z}(t))_\sigma dt + \nu \int_{T_1}^{T_1'} |\mathbf{u}_n(t)|_{1,2} |\mathbf{z}(t)|_{1,2} dt \\ + \int_{T_1}^{T_1'} |\mathbf{g}(t)|_{0,2} |\mathbf{z}(t)|_{0,2} dt + \frac{1}{2} |\mathbf{u}_n(T_1)|_{0,2}^2 + \frac{L_1^2}{2\nu} \int_{T_1}^{T_1'} |\mathbf{g}(t)|_{0,2}^2 dt. \end{aligned} \quad (3.4)$$

Note that

$$\nu \int_{T_1}^{T_1'} |\mathbf{u}_n|_{1,2} |\mathbf{z}|_{1,2} dt \leq \nu \|\mathbf{u}_n\|_{L^2(0,T; \mathbf{V}_\sigma(\Omega))} \|\mathbf{z}\|_{L^2(T_1, T_1'; \mathbf{V}_\sigma(\Omega'))}, \quad (3.5)$$

$$\begin{aligned} \int_{T_1}^{T_1'} |\mathbf{g}|_{0,2} |\mathbf{z}|_{0,2} dt &\leq \|\mathbf{g}\|_{L^2(0,T; \mathbf{H}_\sigma(\Omega))} \|\mathbf{z}\|_{L^2(T_1, T_1'; \mathbf{H}_\sigma(\Omega'))} \\ &\leq L_1 \|\mathbf{g}\|_{L^2(0,T; \mathbf{H}_\sigma(\Omega))} \|\mathbf{z}\|_{L^2(T_1, T_1'; \mathbf{V}_\sigma(\Omega'))}. \end{aligned} \quad (3.6)$$

Besides, as $\operatorname{div} \mathbf{u}_n = \operatorname{div} \mathbf{z} = 0$, we have

$$\begin{aligned} (\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{z}(t))_\sigma &\leq \left| \sum_{k,j=1}^3 \int_{\Omega} u_n^{(k)}(x, t) \frac{\partial u_n^{(j)}(x, t)}{\partial x_k} z^{(j)}(x, t) dx \right| \\ &\leq \left| \sum_{k,j=1}^3 \int_{\Omega} u_n^{(k)}(x, t) \frac{\partial z_n^{(j)}(x, t)}{\partial x_k} u_n^{(j)}(x, t) dx \right| \leq 9 |\mathbf{u}_n(t)|_{0,2} |\mathbf{u}_n(t)|_{0,4} |\mathbf{z}(t)|_{1,4}, \end{aligned}$$

so that

$$\int_{T_1}^{T'_1} (\mathbf{G}(t, \mathbf{u}_n(t)), \mathbf{z}(t))_\sigma dt \leq 9L_3 \|\mathbf{u}_n\|_{L^\infty(0,T;\mathbf{H}_\sigma(\Omega))} \|\mathbf{u}_n\|_{L^2(0,T;\mathbf{V}_\sigma(\Omega))} \|\mathbf{z}\|_{L^2(T_1,T'_1;\mathbf{W}_\sigma(\Omega'))} \quad (3.7)$$

Put (3.5)–(3.7) into (3.4) and neglect the positive terms at the left hand side. Then from (2.8) we obtain for all \mathbf{z} satisfying (3.3) and all $n > N$

$$\begin{aligned} - \int_{T_1}^{T'_1} (\mathbf{u}'_n(t), \mathbf{z}(t))_\sigma dt &\leq M_0 + M_1 \|\mathbf{z}\|_{L^2(T_1,T'_1;\mathbf{V}_\sigma(\Omega'))} + M_2 \|\mathbf{z}\|_{L^2(T_1,T'_1;\mathbf{W}_\sigma(\Omega'))} \\ &\leq M_0 + M_3 \|\mathbf{z}\|_{L^2(T_1,T'_1;\mathbf{W}_\sigma(\Omega'))} \\ &\leq M_0 + M_3 T^{\frac{1}{2}} \|\mathbf{z}\|_{C([T_1,T'_1];\mathbf{W}_\sigma(\Omega'))}, \end{aligned} \quad (3.8)$$

where

$$M_1 := (\nu M_0)^{\frac{1}{2}} + L_1 \|\mathbf{g}\|_{L^2(0,T;\mathbf{H}_\sigma(\Omega))}, \quad M_2 := 9L_3 \nu^{-\frac{1}{2}} M_0, \quad M_3 := M_1 L_2^{\frac{1}{2}} + M_2.$$

Since $-\mathbf{z}$ is also a possible test function, we actually have that for all \mathbf{z} satisfying (3.3) and $n > N$

$$\left| \int_{T_1}^{T'_1} (\mathbf{u}'_n(t), \mathbf{z}(t))_\sigma dt \right| \leq M_0 + M_3 T^{\frac{1}{2}} \|\mathbf{z}\|_{C([T_1,T'_1];\mathbf{W}_\sigma(\Omega'))}.$$

Take finally any $\tilde{\mathbf{z}} \in C_0^1(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))$, put

$$\mathbf{z} = \frac{\tilde{\mathbf{z}}}{\|\tilde{\mathbf{z}}\|_{L^\infty(T_1,T'_1;\mathbf{W}_\sigma(\Omega'))}} \cdot \frac{\kappa}{2L_0}$$

into the above inequality, and obtain that

$$\left| \int_{T_1}^{T'_1} (\mathbf{u}_n(t), \tilde{\mathbf{z}}'(t))_\sigma dt \right| = \left| \int_{T_1}^{T'_1} (\mathbf{u}'_n(t), \tilde{\mathbf{z}}(t))_\sigma dt \right| \leq M_\kappa \|\tilde{\mathbf{z}}\|_{L^\infty(T_1,T'_1;\mathbf{W}_\sigma(\Omega'))}$$

with

$$M_\kappa := \frac{2L_0 M_0}{\kappa} + M_3 T^{\frac{1}{2}},$$

for all $\tilde{\mathbf{z}} \in C_0^1(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))$ and all $n > N$. By a classical result on the relationship between weak derivatives and total variation, see e.g. [7; Prop. A.5], this implies that $\mathbf{u}_n \in BV(T_1, T'_1; \mathbf{W}_\sigma^*(\Omega'))$ and its total variation is bounded by M_κ . \square

Lemma 3.2. *Let p satisfy (1.1) and $\kappa > 0$. Let Ω' be an open set in Ω such that $\tilde{\Omega}' \times [T_1, T'_1] \subset \tilde{Q}(p > \kappa)$, where $\tilde{\Omega}'$ is the relative closure of Ω' in Ω . Then, there exists a function $\mathbf{u}_{\Omega'} : [T_1, T'_1] \rightarrow \mathbf{H}_\sigma(\Omega')$, with $\sup_{T_1 \leq t \leq T'_1} \|\mathbf{u}_{\Omega'}(t)\|_{\mathbf{H}_\sigma(\Omega')} \leq M_0$ for the same constant M_0 as in (2.8), such that, for \mathbf{u}_n defined by (2.11),*

$$\int_{\Omega'} \mathbf{u}_n(t) \cdot \boldsymbol{\xi} dx \rightarrow \int_{\Omega'} \mathbf{u}_{\Omega'}(t) \cdot \boldsymbol{\xi} dx, \text{ as } n \rightarrow \infty, \forall \boldsymbol{\xi} \in \mathbf{H}_\sigma(\Omega'), \forall t \in [T_1, T'_1].$$

Moreover, $|\mathbf{u}_{\Omega'}(x, t)| \leq p(x, t)$ for a.e. $x \in \Omega'$ and every $t \in [T_1, T'_1]$ and $\mathbf{u}_{\Omega'} = \mathbf{u}$ a.e. on $\Omega' \times [T_1, T'_1]$.

Proof. The space $\mathbf{W}_\sigma(\Omega')$ is separable. Let X_0 be its countable dense subset. At a first time, fix any $\boldsymbol{\xi} \in X_0$. We consider the sequence of real functions $f_n : [T_1, T'_1] \rightarrow \mathbf{R}$ defined by

$$f_n(t) = (\mathbf{u}_n(t), \boldsymbol{\xi})_\sigma \left(= \int_{\Omega'} \mathbf{u}_n(x, t) \cdot \boldsymbol{\xi}(x) dx \right).$$

Then, by Lemma 3.1, f_n is uniformly bounded in $W^{1,1}(T_1, T'_1)$, so is its total variation. Therefore, it follows from the Helly selection theorem (see e.g. [12; Section 5.2.3]) that there exists a subsequence f_{n_k} of $\{f_n\}$ and a function $f \in BV(T_1, T'_1)$ such that $f_{n_k} \rightarrow f$ pointwise on $[T_1, T'_1]$ and in $L^1(T_1, T'_1)$.

However, the limit function f and the subsequence f_{n_k} depend also on $\boldsymbol{\xi}$, that is, $n_k = n_k(\boldsymbol{\xi})$ and $f(t) = f(t; \boldsymbol{\xi})$. But, since these are countable, by a diagonal argument we choose a subsequence, denoted again by $\{n_k\}$, such that $(\mathbf{u}_{n_k}(t), \boldsymbol{\xi})_\sigma$ converges to $f(t; \boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in X_0$. Furthermore, by density, this convergence holds for all $\boldsymbol{\xi} \in \mathbf{W}_\sigma(\Omega')$. Indeed, given any $\varepsilon > 0$ and any $\tilde{\boldsymbol{\xi}} \in \mathbf{W}_\sigma(\Omega')$, there exists $\boldsymbol{\xi} \in X_0$ such that $|\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}|_{1,4} < \varepsilon$, so that with $M = \limsup |\mathbf{u}_n|_{-1, \frac{4}{3}} (< \infty)$ (cf. (2.8))

$$\left| (\mathbf{u}_{n_k}(t), \boldsymbol{\xi})_\sigma - (\mathbf{u}_{n_k}(t), \tilde{\boldsymbol{\xi}})_\sigma \right| \leq |\mathbf{u}_{n_k}|_{-1, \frac{4}{3}} |\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}|_{1,4} \leq M\varepsilon.$$

This shows that $f(t, \tilde{\boldsymbol{\xi}}) = \lim_{k \rightarrow \infty} (\mathbf{u}_{n_k}(t), \tilde{\boldsymbol{\xi}})_\sigma$ exists, i.e. we can extend $f(t; \cdot)$ to $\mathbf{W}_\sigma(\Omega')$.

Note that $f(t, \cdot)$ is linear as limit of linear functions. Thus, by the Riesz theorem,

$$\exists \tilde{\mathbf{u}} : [T_1, T'_1] \rightarrow \mathbf{W}_\sigma^*(\Omega') \text{ such that } f(t; \boldsymbol{\xi}) = (\tilde{\mathbf{u}}(t), \boldsymbol{\xi})_\sigma, \quad \forall \boldsymbol{\xi} \in \mathbf{W}_\sigma(\Omega').$$

Since $|\mathbf{u}_{n_k}(t)|_{0,2} \leq M_0$ for all $t \in [0, T]$, we have $\tilde{\mathbf{u}}(t) \in \mathbf{H}_\sigma(\Omega')$ for all $t \in [T_1, T'_1]$ and $\sup_{T_1 \leq t \leq T'_1} \|\tilde{\mathbf{u}}(t)\|_{\mathbf{H}_\sigma(\Omega')} \leq M_0$ and $\mathbf{u}_{n_k}(t) \rightarrow \tilde{\mathbf{u}}(t)$ weakly in $\mathbf{H}_\sigma(\Omega')$ as $k \rightarrow \infty$.

We show now that $|\tilde{\mathbf{u}}|$ is bounded a.e. by p . Take any $\varepsilon > 0$ and an integer $k(\varepsilon)$ large enough to have $p_{n_k}(x, t) \leq p(x, t) + \varepsilon$ for all $x \in Q$ and all $k \geq k(\varepsilon)$, cf. (2.1). Then $|\mathbf{u}_{n_k}(x, t)| \leq p_{n_k}(x, t) \leq p(x, t) + \varepsilon$ for a.e. $(x, t) \in \Omega$. We note that the set

$$F_\varepsilon(t) = \{z \in \mathbf{H}_\sigma(\Omega') \mid |z(x)| \leq p(x, t) + \varepsilon \text{ on } \Omega'\}$$

is convex and closed in $\mathbf{H}_\sigma(\Omega')$. It follows from the Mazur lemma (cf. [29; Th.2, Ch.V]) that the weak limit $\tilde{\mathbf{u}}(t)$ of $\mathbf{u}_{n_k}(t)$ in $\mathbf{H}_\sigma(\Omega')$ belongs to $F_\varepsilon(t)$. By arbitrariness of $\varepsilon > 0$, we have $|\tilde{\mathbf{u}}(x, t)| \leq p(x, t)$ for a.e. $x \in \Omega'$.

We finally show that $\tilde{\mathbf{u}} = \mathbf{u}$ a.e. in $\Omega' \times [T_1, T'_1]$. Take $\{E_i\}_{i=1}^N$ a partition of $[T_1, T'_1]$ (family of pairwise disjoint measurable sets covering the interval) and let $\boldsymbol{\zeta}$ be a function of the form

$$\boldsymbol{\zeta}(t) = \sum_{i=1}^N \chi_{E_i}(t) \boldsymbol{\xi}_i \text{ with } \boldsymbol{\xi}_i \in \mathbf{W}_\sigma(\Omega').$$

Then, by definition of $\tilde{\mathbf{u}}$ and since the sum is finite,

$$\int_{T_1}^{T'_1} (\mathbf{u}_{n_k}(t), \boldsymbol{\zeta}(t))_{\sigma} dt \rightarrow \int_{T_1}^{T'_1} (\tilde{\mathbf{u}}(t), \boldsymbol{\zeta}(t))_{\sigma} dt.$$

On the other hand, since $\mathbf{u}_n \rightarrow \mathbf{u}$ weakly in $L^2(0, T, \mathbf{V}_{\sigma}(\Omega'))$, cf. (2.11), and as $\mathbf{W}_{\sigma}(\Omega') \hookrightarrow \mathbf{V}_{\sigma}(\Omega')$,

$$\int_{T_1}^{T'_1} (\mathbf{u}_n(t), \boldsymbol{\zeta}(t))_{\sigma} dt \rightarrow \int_{T_1}^{T'_1} (\mathbf{u}(t), \boldsymbol{\zeta}(t))_{\sigma} dt.$$

Consequently, by density of $\mathbf{W}_{\sigma}(\Omega')$ in $\mathbf{H}_{\sigma}(\Omega')$, $\tilde{\mathbf{u}} = \mathbf{u}$ a.e. on $\Omega' \times [T_1, T'_1]$. By uniqueness of the limit, all convergences stated above are valid for all the sequences, without extracting any subsequence. We obtain the statement of the lemma, where $\mathbf{u}_{\Omega'} := \tilde{\mathbf{u}}$ is the required function. \square

Corollary 3.1. *Assume (1.1). Let $\Omega(t, 0) := \{x \in \Omega \mid p(x, t) > 0\}$ for each $t \in [0, T]$. Then there exists a function $\bar{\mathbf{u}} : [0, T] \rightarrow \mathbf{H}_{\sigma}(\Omega)$, with $\sup_{t \in [0, T]} |\bar{\mathbf{u}}(t)|_{0,2} \leq M_0$, such that $\bar{\mathbf{u}}(t) = \mathbf{u}_{\Omega'}(t)$ in $\mathbf{H}_{\sigma}(\Omega')$ for any open and relatively compact subset Ω' of $\Omega(t, 0)$ and any $t \in [0, T]$, where M_0 is the same constant as in (2.8) and $\mathbf{u}_{\Omega'}(t)$ is the function constructed in Lemma 3.2, corresponding to Ω' . Moreover, for \mathbf{u}_n defined by (2.11),*

$$\mathbf{u}_n(t) \rightarrow \bar{\mathbf{u}}(t) \text{ weakly in } \mathbf{H}_{\sigma}(\Omega), \quad |\bar{\mathbf{u}}(x, t)| \leq p(x, t) \text{ for a.e. } x \in \Omega, \quad \forall t \in [0, T], \quad (3.9)$$

and $\bar{\mathbf{u}} = \mathbf{u}$ a.e. on Q .

Proof. For each $t \in [0, T]$, by (1.1), the set $\Omega(t, 0)$ is a countable union of non-decreasing, open and relatively compact subsets Ω'_i in Ω , $i \in \mathbf{N}$, such that $p(x, t) > \kappa_i$ on Ω'_i , $\kappa_i \rightarrow 0$. By Lemma 3.2, the limit $\lim_{i \rightarrow \infty} \int_{\Omega'_i} \mathbf{u}_{\Omega'_i}(t) \cdot \boldsymbol{\xi} dx$ exists for all $\boldsymbol{\xi} \in \mathbf{H}_{\sigma}(\Omega)$. This limit is linear and bounded with respect to $\boldsymbol{\xi}$ in $\mathbf{H}_{\sigma}(\Omega)$ and determines a unique element $\bar{\mathbf{u}}(t)$ in $\mathbf{H}_{\sigma}(\Omega)$ by the formula

$$\int_{\Omega} \bar{\mathbf{u}}(t) \cdot \boldsymbol{\xi} dx := \lim_{i \rightarrow \infty} \int_{\Omega'_i} \mathbf{u}_{\Omega'_i}(t) \cdot \boldsymbol{\xi} dx, \quad \forall \boldsymbol{\xi} \in \mathbf{H}_{\sigma}(\Omega), \quad \forall t \in [0, T].$$

We also have, by Lemma 3.2, $\sup_{t \in [0, T]} |\bar{\mathbf{u}}(t)|_{0,2} \leq M_0$ and $\bar{\mathbf{u}}(x, t) = 0$ for a.e. $x \in \Omega$ such that $p(x, t) = 0$. Indeed, taking $\bar{\mathbf{u}}(t)$ as $\boldsymbol{\xi}$ above, we get

$$\int_{\Omega} |\bar{\mathbf{u}}(x, t)|^2 dx = \int_{\Omega(t, 0)} |\bar{\mathbf{u}}(x, t)|^2 dx,$$

which implies that $|\bar{\mathbf{u}}(x, t)| = 0$ for a.e. $x \in \Omega - \Omega(t, 0) = \{x \in \Omega \mid p(x, t) = 0\}$. Thus (3.9) is obtained.

Finally, we show that $\bar{\mathbf{u}} = \mathbf{u}$ a.e. on Q . To do so, for any $\boldsymbol{\zeta} \in L^2(0, T; \mathbf{H}_{\sigma}(\Omega))$, we observe from (3.9) and Lemma 3.2 that

$$\int_0^T (\bar{\mathbf{u}}(t), \boldsymbol{\zeta}(t))_{\sigma} dt = \lim_{i \rightarrow \infty} \int_0^T \int_{\Omega'_i} \mathbf{u}_{\Omega'_i} \cdot \boldsymbol{\zeta} dx dt = \int_0^T (\mathbf{u}(t), \boldsymbol{\zeta}(t))_{\sigma} dt,$$

which implies that $\bar{\mathbf{u}} = \mathbf{u}$ a.e. on Q . Thus $\bar{\mathbf{u}}$ is the required function. \square

By virtue of Corollary 3.1, we may identify the function \mathbf{u} with $\bar{\mathbf{u}}$; namely \mathbf{u} is a function defined for every $t \in [0, T]$ with values in $\mathbf{H}_\sigma(\Omega)$.

Corollary 3.2. *Let Ω' be an open set in Ω and $0 \leq T_1 < T'_1 \leq T$. Assume that $\tilde{\Omega}' \times [T_1, T'_1] \subset \hat{Q}(p = \infty)$. Then $\mathbf{u} \in W^{1,2}(T_1, T'_1; \mathbf{W}_\sigma^*(\Omega'))$ and hence \mathbf{u} is absolutely continuous as a function from $[T_1, T'_1]$ into $\mathbf{W}_\sigma^*(\Omega')$.*

Proof. Let \mathbf{z} be any function in $C_0^1(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))$ and take a (large) positive number κ and a positive integer $n(\kappa)$ so that

$$\|\mathbf{z}\|_{C([T_1, T'_1]; \mathbf{W}_\sigma(\Omega'))} < \frac{\kappa}{2L_0}$$

and

$$p_n(x, t) > \frac{\kappa}{2}, \quad \forall (x, t) \in \Omega' \times [T_1, T'_1], \quad \forall n \geq n(\kappa).$$

Then we observe that

$$|\mathbf{z}(x, t)| \leq L_0 |\mathbf{z}|_{1,4} < \frac{\kappa}{2} < p_n(x, t), \quad \forall (x, t) \in \Omega' \times [T_1, T'_1], \quad \forall n \geq n(\kappa).$$

Therefore, just as in the proof of Lemma 3.1, we have (3.8) and

$$\left| \int_{T_1}^{T'_1} (\mathbf{u}_n(t), \mathbf{z}'(t))_\sigma dt \right| = \left| \int_{T_1}^{T'_1} (\mathbf{u}'_n(t), \mathbf{z}(t))_\sigma dt \right| \leq M_0 + M_3 \|\mathbf{z}\|_{L^2(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))}.$$

The right hand side of the last inequality can be changed to $(M_0 + M_3) \|\mathbf{z}\|_{L^2(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))}$ with the same trick as in the proof of Lemma 3.1: whenever $\|\mathbf{z}\|_{L^2(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))} < 1$, we put

$$\tilde{\mathbf{z}} = \frac{\mathbf{z}}{\|\mathbf{z}\|_{L^2(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))}}$$

above. Finally, letting $n \rightarrow \infty$, we obtain

$$\left| \int_{T_1}^{T'_1} (\mathbf{u}(t), \mathbf{z}'(t))_\sigma dt \right| \leq (M_0 + M_3) \|\mathbf{z}\|_{L^2(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))}$$

for all $\mathbf{z} \in C_0^1(T_1, T'_1; \mathbf{W}_\sigma(\Omega'))$. This shows that $\mathbf{u} \in W^{1,2}(T_1, T'_1; \mathbf{W}_\sigma^*(\Omega'))$, whence \mathbf{u} is absolutely continuous as a function from $[T_1, T'_1]$ into $\mathbf{W}_\sigma^*(\Omega')$. \square

Lemma 3.3. *Assume (1.1). Take $\kappa > 0$ and let Ω' be an open set in Ω such that $\tilde{\Omega}' \times [T_1, T'_1] \subset \hat{Q}(p > \kappa)$, where $\tilde{\Omega}'$ is the relative closure of Ω' in Ω . Then, for \mathbf{u}_n defined by (2.11), we have $\mathbf{u}_n \rightarrow \mathbf{u}$ (strongly) in $L^2(T_1, T'_1; \mathbf{H}_\sigma(\Omega'))$ as $n \rightarrow \infty$.*

Proof. On account of the Aubin's compactness lemma (see [20; Lemma 5.1]), for any $\varepsilon > 0$ there exists a positive constant A_ε such that

$$|\mathbf{z}|_{0,2}^2 \leq \varepsilon |\mathbf{z}|_{1,2}^2 + A_\varepsilon |\mathbf{z}|_{-1, \frac{4}{3}}^2, \quad \forall \mathbf{z} \in \mathbf{V}_\sigma(\Omega').$$

Thus,

$$\int_{T_1}^{T'_1} |\mathbf{u}_n - \mathbf{u}|_{0,2}^2 dt \leq \varepsilon \int_{T_1}^{T'_1} |\mathbf{u}_n - \mathbf{u}|_{1,2}^2 dt + A_\varepsilon \int_{T_1}^{T'_1} |\mathbf{u}_n - \mathbf{u}|_{-1, \frac{4}{3}}^2 dt.$$

By Lemma 3.2, the last term tends to 0. Therefore, by (2.8) of Proposition 2.1 we derive from the above inequality

$$\limsup_{n \rightarrow \infty} \int_{T_1}^{T'_1} |\mathbf{u}_n - \mathbf{u}|_{0,2}^2 dt \leq \frac{2\varepsilon}{\nu} M_0.$$

Since ε is arbitrary, this gives the statement of the lemma. \square

4 The proof of Theorem 1.1

In all this section, \mathbf{u}_n is the sequence defined by (2.11). We identify the function \mathbf{u} with $\bar{\mathbf{u}}$ constructed in Corollary 3.1; hence we have:

$$\mathbf{u}_n(t) \rightarrow \mathbf{u}(t) \text{ weakly in } \mathbf{H}_\sigma(\Omega), \quad \forall t \in [0, T], \quad (4.1)$$

$$|\mathbf{u}(x, t)| \leq p(x, t), \quad \text{a.e. } x \in \Omega, \quad \forall t \in [0, T]. \quad (4.2)$$

Furthermore, we have the following lemma.

Lemma 4.1. $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^2(0, T; \mathbf{H}_\sigma(\Omega))$ as $n \rightarrow \infty$.

Proof. Let ε be any positive number and consider $\hat{Q}(p \leq \kappa)$ with $\kappa := (\frac{\varepsilon}{18T|\Omega|})^{\frac{1}{2}}$, $|\Omega|$ being the volume of Ω . By (2.1), for a large integer $n_1(\varepsilon)$ we have

$$|\mathbf{u}_n(x, t)| \leq p_n(x, t) \leq p(x, t) + \kappa \leq 2\kappa, \quad \text{for a.e. } (x, t) \in \hat{Q}(p \leq \kappa), \quad \forall n > n_1(\varepsilon).$$

Therefore, using (4.2) noted above,

$$\int \int_{\hat{Q}(p \leq \kappa)} |\mathbf{u}_n(x, t) - \mathbf{u}(x, t)|^2 dx dt \leq 9T |\Omega| \kappa^2 = \frac{\varepsilon}{2}. \quad (4.3)$$

Next, consider $\hat{Q}(p > \kappa)$. Take any $\kappa' \in (0, \kappa)$. Note that $\hat{Q}(p > \kappa) \subset \hat{Q}(p > \kappa')$ and that by (1.1) we can find a finite number of cylindrical domains of the form $\Omega_i \times [\tau_i, \tau'_i]$, $i = 1, 2, \dots, N$, such that $\tilde{\Omega}_i$ (the relative closure of Ω_i in Ω) is contained in Ω and

$$\hat{Q}(p > \kappa) \subset \bigcup_{i=1}^N \Omega_i \times [\tau_i, \tau'_i] \subset \bigcup_{i=1}^N \tilde{\Omega}_i \times [\tau_i, \tau'_i] \subset \hat{Q}(p > \kappa').$$

Indeed, for any $(x, t) \in \hat{Q}(p > \kappa)$ there exists an open set $\Omega(x, t) \subset \Omega$ with $\tilde{\Omega}(x, t) \subset \Omega$, and there exist $\tau := \tau(x, t)$, $\tau' := \tau'(x, t)$ with $\tau < \tau'$ such that $t \in [\tau, \tau']$ and $\Omega(x, t) \times [\tau, \tau'] \subset \hat{Q}(p > \kappa')$. We take a finite covering of $\hat{Q}(p > \kappa)$ from this family. For such a finite covering $\Omega_i \times [\tau_i, \tau'_i]$, $i = 1, 2, \dots, N$, it follows from Lemma 3.3 that there is a positive integer $n_2(\varepsilon)$ such that for all $n > n_2(\varepsilon)$

$$\int_{\hat{Q}(p > \kappa)} |\mathbf{u}_n(x, t) - \mathbf{u}(x, t)|^2 dx dt \leq \sum_{i=1}^N \|\mathbf{u}_n - \mathbf{u}\|_{L^2(\tau_i, \tau'_i; \mathbf{H}_\sigma(\Omega_i))}^2 \leq \frac{\varepsilon}{2}. \quad (4.4)$$

Summing (4.3) and (4.4), we obtain $\int_Q |\mathbf{u}_n - \mathbf{u}|^2 dx dt \leq \varepsilon$. The lemma is proved. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: We consider the approximate problems $P(p_n; \mathbf{g}, \mathbf{u}_{0n})$ defined by Definition 2.1 with p_n satisfying (2.1) and $\mathbf{u}_{0n} = (1 - \hat{\delta}_n)^+ \mathbf{u}_0$, with $\hat{\delta}_n$ as in Proposition 2.1. Let \mathbf{v} be any test function from $\mathcal{K}(p)$. Then, for some positive constant δ we have $\text{supp}(\mathbf{v}) \subset \hat{Q}(p > \delta)$, so that by Lemma 2.3 its approximate sequence

$$\mathbf{z}_n(t) = (1 - \delta_n)^+ \mathbf{v}(t) \in K(p_n; t), \quad \forall t \in [0, T], \quad \forall n,$$

satisfies

$$\text{supp}(\mathbf{z}_n) \subset \hat{Q}(p > \delta), \quad \mathbf{z}_n \rightarrow \mathbf{v} \quad \text{in } C^1([0, T]; \mathbf{W}_\sigma(\Omega)).$$

We can now take \mathbf{z}_n as test function in (2.4) to get for all large n with $\delta_n < 1$ that

$$\begin{aligned} & (\mathbf{u}'_n(\tau), \mathbf{u}_n(\tau) - \mathbf{v}(\tau))_\sigma + \nu \langle \mathbf{u}_n(\tau), \mathbf{u}_n(\tau) - \mathbf{v}(\tau) \rangle_\sigma \\ & \quad + (\mathbf{G}(\tau, \mathbf{u}_n(\tau)), \mathbf{u}_n(\tau) - \mathbf{v}(\tau))_\sigma \\ & \leq (\mathbf{g}(\tau), \mathbf{u}_n(\tau) - \mathbf{v}(\tau))_\sigma - \delta_n Y_n(\tau) - \delta_n Z_n(\tau), \quad \text{a.e. } \tau \in [0, T], \end{aligned} \tag{4.5}$$

where

$$Y_n(\tau) := (\mathbf{u}'_n(\tau), \mathbf{v}(\tau))_\sigma$$

and

$$Z_n(\tau) := \nu \langle \mathbf{u}_n(\tau), \mathbf{v}(\tau) \rangle_\sigma + (\mathbf{G}(\tau, \mathbf{u}_n(\tau)), \mathbf{v}(\tau))_\sigma - (\mathbf{g}(\tau), \mathbf{v}(\tau))_\sigma.$$

Here we note that

$$\left| \int_0^t Y_n(\tau) d\tau \right| \leq |(\mathbf{u}_0, \mathbf{v}(0))_\sigma| + |(\mathbf{u}_n(t), \mathbf{v}(t))_\sigma| + T \|\mathbf{u}_n\|_{L^\infty(0, T; \mathbf{H}_\sigma(\Omega))} \|\mathbf{v}'\|_{C([0, T]; \mathbf{H}_\sigma(\Omega))}$$

and this is uniformly bounded on $[0, T]$. Moreover,

$$\begin{aligned} \int_0^t |Z_n(\tau)| d\tau & \leq \nu \|\mathbf{u}_n\|_{L^2(0, T; \mathbf{V}_\sigma)} \|\mathbf{v}\|_{L^2(0, T; \mathbf{V}_\sigma)} \\ & \quad + 9L_3 \|\mathbf{u}_n\|_{L^2(0, T; \mathbf{H}_\sigma)} \|\mathbf{u}_n\|_{L^2(0, T; \mathbf{V}_\sigma)} \|\mathbf{v}\|_{C([0, T]; \mathbf{W}_\sigma(\Omega))} \\ & \quad + \|\mathbf{g}\|_{L^2(0, T; \mathbf{H}_\sigma)} \|\mathbf{v}\|_{L^2(0, T; \mathbf{H}_\sigma(\Omega))}, \quad \forall t \in [0, T] \end{aligned}$$

and the right hand side is bounded in n on account of estimate (2.8). Therefore, we have

$$\delta_n \int_0^t \{Y_n(\tau) + Z_n(\tau)\} d\tau \rightarrow 0 \quad \text{uniformly on } [0, T]. \tag{4.6}$$

After integration of (4.5) in time over $[0, t]$, use the integration by parts in the resultant and recall that $(\mathbf{G}(\tau, \mathbf{u}_n(\tau)), \mathbf{u}_n(\tau)) = 0$. Then,

$$\begin{aligned} & \int_0^t (\mathbf{v}', \mathbf{u}_n - \mathbf{v})_\sigma d\tau + \frac{1}{2} |\mathbf{u}_n(t) - \mathbf{v}(t)|_{0,2}^2 \\ & \quad + \nu \int_0^t \langle \mathbf{u}_n, \mathbf{u}_n - \mathbf{v} \rangle_\sigma d\tau - \int_0^t \langle \mathbf{G}(\tau, \mathbf{u}_n), \mathbf{v} \rangle_\sigma d\tau \\ & \leq \int_0^t (\mathbf{g}, \mathbf{u}_n - \mathbf{v})_\sigma d\tau + \frac{1}{2} |\mathbf{u}_n(0) - \mathbf{v}(0)|_{0,2}^2 - \delta_n \int_0^t \{Y_n + Z_n\} d\tau. \end{aligned} \tag{4.7}$$

Now we pass to the limit $n \rightarrow \infty$ in (4.7). The first term of the left hand side and the others are bounded from below by the respective terms with the limit \mathbf{u} , since $\mathbf{u}_n \rightarrow \mathbf{u}$ weakly in $L^2(0, T; \mathbf{V}_\sigma(\Omega))$ and $\mathbf{u}_n(t) \rightarrow \mathbf{u}(t)$ weakly in $\mathbf{H}_\sigma(\Omega)$ for every $t \in [0, T]$ by (4.1). This also allows to pass to the limit in the first term of the right hand side. The initial condition is chosen so that $\mathbf{u}_n(0) = \mathbf{u}_{0n} \rightarrow \mathbf{u}_0$ in $\mathbf{W}_\sigma(\Omega)$. As for the nonlinear term, we observe that

$$\begin{aligned} & \int_0^t (\mathbf{G}(\tau, \mathbf{u}_n) - (\mathbf{G}(\tau, \mathbf{u}(\tau)), \mathbf{v}))_\sigma d\tau \\ &= \sum_{k,j=1}^3 \int_0^t \int_\Omega \left\{ v^{(j)} (u_n^{(k)} - u^{(k)}) \frac{\partial u_n^{(j)}}{\partial x_k} + v^{(j)} u^{(k)} \frac{\partial (u_n^{(j)} - u^{(j)})}{\partial x_k} \right\} dx d\tau \rightarrow 0 \end{aligned}$$

by virtue of strong convergence $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^2(0, T; \mathbf{H}_\sigma(\Omega))$ (cf. Lemma 4.1), combined with the weak convergence in $L^2(0, T; \mathbf{V}_\sigma(\Omega))$. Consequently, for any $\mathbf{v} \in \mathcal{K}(p)$ it follows from (4.6) and (4.7) that

$$\begin{aligned} & \int_0^t (\mathbf{v}', \mathbf{u} - \mathbf{v})_\sigma d\tau + \frac{1}{2} |\mathbf{u}(t) - \mathbf{v}(t)|_\sigma^2 + \nu \int_0^t \langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle_\sigma d\tau \\ & \quad + \int_0^t \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \nabla (\mathbf{u} - \mathbf{v}) dx d\tau \\ & \leq \int_0^t (\mathbf{g}, \mathbf{u} - \mathbf{v})_\sigma d\tau + \frac{1}{2} |\mathbf{u}_0 - \mathbf{v}(0)|_\sigma^2, \end{aligned}$$

i.e. \mathbf{u} solves (1.2). With (4.2), the condition (ii) of Definition 1.1 is satisfied.

It remains to show (i). As for the initial condition, $\mathbf{u}(0) = \mathbf{u}_0$ since \mathbf{u}_{0n} converges strongly to \mathbf{u}_0 in $\mathbf{W}_\sigma(\Omega)$. Let us show that $t \mapsto (\mathbf{u}(t), \mathbf{v}(t))_\sigma$ is of bounded variation on $[0, T]$ for every $\mathbf{v} \in \mathcal{K}(p)$. For simplicity, we denote

$$m_v = \|\mathbf{v}(t)\|_{L^\infty(0, T; \mathbf{H}_\sigma(\Omega))}, \quad M_v = \|\mathbf{v}(t)\|_{C([0, T]; \mathbf{W}_\sigma(\Omega))}.$$

Since $\text{supp}(\mathbf{v}) \subset \hat{Q}(p > \kappa)$ for a certain $\kappa > 0$, we can find a finite covering $\Omega_i \times [T_i, T'_i]$, $i = 1, 2, \dots, N$, of $\text{supp}(\mathbf{v})$ such that

$$\text{supp}(\mathbf{v}) \subset \bigcup_{i=1}^N \Omega_i \times [T_i, T'_i] \subset \bigcup_{i=1}^N \tilde{\Omega}_i \times [T_i, T'_i] \subset \hat{Q}(p > \kappa).$$

For each i , it follows from Lemma 3.1 that the restriction of \mathbf{u} to $[T_i, T'_i] \times \Omega_i$ is of bounded variation as a function from $[T_i, T'_i]$ into $\mathbf{W}_\sigma^*(\Omega_i)$; we denote by $V_i(\mathbf{u})$ its total variation. Now, for any $s, t \in [T_i, T'_i]$ we observe that

$$\begin{aligned} |(\mathbf{u}(t), \mathbf{v}(t))_\sigma - (\mathbf{u}(s), \mathbf{v}(s))_\sigma| &\leq |(\mathbf{u}(t) - \mathbf{u}(s), \mathbf{v}(t))_\sigma + (\mathbf{u}(s), \mathbf{v}(t) - \mathbf{v}(s))_\sigma| \\ &\leq M_v |\mathbf{u}(t) - \mathbf{u}(s)|_{\mathbf{W}_\sigma^*(\Omega_i)} + m_u |\mathbf{v}(t) - \mathbf{v}(s)|_{\mathbf{H}_\sigma(\Omega)}. \end{aligned}$$

The total variation of $t \mapsto (\mathbf{u}(t), \mathbf{v}(t))_\sigma$ on the interval $[T_i, T'_i]$ is bounded by $M_v V_i(\mathbf{u}) + m_u \int_{T_i}^{T'_i} |\mathbf{v}'|_{0,2} dt$. Therefore, the total variation on the whole interval $[0, T]$ is not larger than

$$M_v \sum_{i=1}^N V_i(\mathbf{u}) + m_u \int_0^T |\mathbf{v}'|_{0,2} dt (< \infty).$$

The proof of Theorem 1.1 is now complete. \square

Remark 4.1. In particular, if the support of the test function \mathbf{v} is contained in $\hat{Q}(p = \infty)$, then it follows from Corollary 3.1 that the function $t \mapsto (\mathbf{u}(t), \mathbf{v}(t))_\sigma$ is absolutely continuous on $[0, T]$ and $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$ on $\{x \in \Omega \mid p(x, 0) = \infty\}$ pointwise.

Remark 4.2. Assume that $\mathbf{g} \equiv 0$, i.e. no external forces are present. In this case, if at time t_0 the whole region Ω is blocked by a total obstacle, i.e. $p(x, t_0) = 0$ for all $x \in \Omega$, then the flow vanishes starting from the moment t_0 : $\mathbf{v} \equiv 0$ in $[t_0, T] \times \Omega$. In other words, if the obstacle grows to the whole region at some time t_0 , then it blocks the flow efficiently, even if the obstacle itself diminishes afterwards. In fact, since $\mathbf{v}(\cdot, t_0) \equiv 0$ in Ω , it follows that $\mathbf{v} \equiv 0$ is the trivial solution of the Navier–Stokes equation on $(t_0, T) \times \Omega$ and this is the solution of the variational inequality of the Navier–Stokes type, which can be constructed in our approximate procedure, too.

References

1. H. Abels, Longtime behavior of solutions of a Navier–Stokes/Cahn–Hilliard system, pp. 9–19 in *Nonlocal and abstract parabolic equations and their applications*, Banach Center Publ. **Vol. 86**, Polish Acad. Sci., Inst. Math., Warsaw, 2009.
2. W. H. Alt and I. Pawlow, Existence of solutions for non-isothermal phase separation, *Adv. Math. Sci. Appl.*, **1** (1992), 319–409.
3. M. Biroli, Sur l’inéquation d’évolution de Navier–Stokes. *C. R. Acad. Sci. Paris Ser. A-B*, 275 (1972), A365–A367.
4. M. Biroli, Sur inéquation d’évolution de Navier–Stokes. Nota I, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 52 (1972), 457–460.
5. M. Biroli, Sur inéquation d’évolution de Navier–Stokes. Nota II, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 52 (1972), 591–598.
6. M. Biroli, Sur inéquation d’évolution de Navier–Stokes. Nota III, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 52 (1972), 811–820.
7. H. Brézis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les espaces de Hilbert*, Math. Studies 5, North-Holland, Amsterdam, 1973.
8. M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, **Vol. 121**, Springer-Verlag, Berlin–Heidelberg–New York, 1996.
9. P. Colli, N. Kenmochi and M. Kubo, A phase-field model with temperature dependent constraint, *J. Math. Anal. Appl.*, **256** (2001), 668–685.
10. A. Damlamian, Some results on the multi-phase Stefan problem, *Comm. Partial Differential Equations*, **2** (1977), 1017–1044.

11. H.J. Eberl, D.F. Parker and M.C.M. van Loosdrecht, A new deterministic spatio-temporal continuum model for biofilm development, *Computational and Mathematical Methods in Medicine*, **3** (2001), 161–175.
12. L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton–London–New York–Washington, D.C., 1992.
13. T. Fukao and N. Kenmochi, Variational inequality for the Navier-Stokes equations with time-dependent constraint, *Gakuto Internat. Ser. Math. Sci. Appl.*, Vol. 34 (2011), 87–102.
14. T. Fukao and N. Kenmochi, Quasi-variational inequalities approach to heat convection problems with temperature dependent velocity constraint, *Discrete Contin. Dyn. Syst.*, **35** (2015), 2523–2538.
15. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin–Heidelberg–New York, 1998.
16. H. Inoue and M. Ôtani, Periodic problems for heat convection equations in non-cylindrical domains, *Funkc. Ekvac.*, **40** (1997), 19–39.
17. N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, *Bull. Fac. Edu., Chiba Univ.*, **30** (1981), 1–87.
18. N. Kenmochi and M. Niezgódka, Viscosity approach to modelling non-isothermal diffusive phase separation, *Jpn. J. Ind. Appl. Math.*, **13** (1996), 135–169.
19. M. Kubo, A. Ito and N. Kenmochi, Non-isothermal phase separation models: weak well-posedness and global estimates, *Gakuto Internat. Ser. Math. Sci. Appl.*, **Vol. 14** (2000), 311–323.
20. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier–Villars, Paris, 1969.
21. Y. Murase and A. Ito, Mathematical model for the process of brewing Japanese sake and its analysis, *Adv. Math. Sci. Appl.* **23** (2013), 297–317.
22. M. Ôtani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, *J. Differential Equations*, **46** (1982), 268–299.
23. M. Peszyńska, A. Trykozko, G. Iltis and S. Schlueter, Biofilm growth in porous media: Experiments, computational modeling at the porescale, and upscaling, *Advances in Water Resources*, 1–14, 2015.
24. G. Prouse, On an inequality related to the motion, in any dimension, of viscous, incompressible fluids. Note I, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 67 (1979), 191–196.
25. G. Prouse, On an inequality related to the motion, in any dimension, of viscous, incompressible fluids. Note II, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 67 (1979), 282–288.

26. K. Shirakawa, A. Ito, N. Yamazaki and N. Kenmochi, Asymptotic stability for evolution equations governed by subdifferentials, pp. 287-310 in *Recent Development in Domain Decomposition Methods and Flow Problems*, Gakuto Internat. Ser. Math. Sci. Appl., **Vol. 11**, Gakkōtoshō, Tokyo, 1998.
27. R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.
28. Y. Yamada, On nonlinear evolution equations generated by the subdifferentials, J. Fac. Sci. Univ. Tokyo, Sect. IA, **23**(1976), 491–515.
29. K. Yosida, *Functional Analysis* (Sixth edition), Springer-Verlag, Berlin–Heidelberg–New York, 1980.